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The time-optimal transfer of a perturbed dynamical object to a given position☆

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ABSTRACT

The problem of transferring a dynamical object (a point mass) of arbitrary dimensional to a required position in a time-optimal manner by means of a force of limited modulus is solved. The velocity of the object at the final instant is not specified. It is assumed that an arbitrary known perturbation, with a magnitude strictly less than that of the control, acts on the controlled system. For clarity when analysing the optimal controlled motion, considerable attention is paid to the case of a steady perturbation. A constructive procedure for finding the optimal response time and the control is developed for arbitrary permissible values of the governing parameters. The Bellman function and the feedback control are constructed over the whole of the phase space. The structural properties of the solution are established and an asymptotic analysis is carried out by small-parameter methods. The extremal directions of the perturbation vector and the corresponding response time and optimal control are found. A modification of the time-optimal problem to the case of a non-stationary perturbation is presented and the basic properties of the optimal solution are investigated.

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1. Formulation of the problem

The controlled motion of a point mass of constant mass m under the action of a force of limited modulus $P(t)| \le P_0$ is considered. The direction of this force is arbitrary in the Euclidean space E^n, $n \ge 1$. It is postulated that, together with the control force P, there is also a known perturbation F, $|F| < P_0$ in the system. It is assumed that the problem of controlling the motion of a multidimensional dynamical object is described by the relations

$$\begin{aligned} &\dot{x} = \upsilon, \quad m\dot{\upsilon} = P + F; \quad x(0) = x^0, \quad \dot{\upsilon}(0) = \upsilon^0 \\ &x(T) = x^T, \quad |\upsilon(T)| < \infty; \quad x, \upsilon, P, F \in E^n, \quad 1 \le n < \infty \\ &J[P] = T \to \min_P, \quad |P| \le P_0, \quad |F| \le F_0 < P_0 \end{aligned} \tag{1.1}$$

It is required to transfer the point to a specified position $x^T \ne x^0$ in accordance with relations (1.1) in the least time T, where the value of the velocity $\upsilon(T)$ is not specified. The initial data x^0, υ^0 are assumed to be arbitrary.

The formulation of the time-optimal problem (1.1) corresponds to the case $F = \text{const}$, which is of interest at the initial stage owing to its simplicity and clarity. The much more general situation of a variable perturbation $F(t)$ requires a reformulation of the time-optimal problem of the type (1.1) (see Section 6).

For greater generality in formulating problem (1.1), we can assume that the values of the control force P are bounded by the non-degenerate ellipsoid: $|QP| \le P_0$, $\det Q \ne 0$. By means of the non-singular transformation of the vectors x, υ, P and F

$$x' = Qx, \upsilon' = Q\upsilon, \ P' = QP, \ F' = QF, \ |F'| \le F_0' < P_0$$

the problem reduces to the form (1.1). Vector operations in the space E^n remain as before.

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Hence, the time-optimal problem is further considered in the initial formulation (1.1). It is required to construct the optimal control in the open-loop form $P^* = u_p(t, x_0, \upsilon^0; x^T, m, P_0, F)$ and in the feedback form $P^* = u_s(x, \upsilon^0; \upsilon^T, m, P_0, F)$, the corresponding phase trajectory $x^* = x(t, x^0, \upsilon^0; x^T, m, P, F)$ $\upsilon^* = \upsilon(t, x^0, \upsilon^0; x^T, m, P_0, F)$ and, also, to calculate the optimal response time $T^* = T_p(x^0, \upsilon^0; x^T, m, P_0, F)$ and the Bellman function $T^* = T_s(x, \upsilon; x^T, m, P_0, F)$.

Note that the system contains $4n+2$ constant parameters: the n-vectors x^0, x^T, υ^0, F and the scalars m and P_0. Their number can be reduced to $n+2$ by translating of the coordinate system $x' = x - x^T$ and the introducing of the new argument $t' = (P_0/m)^{1/2}t$, the new control $u = P/P_0$ and the new perturbation $F = F/P_0$. In order for the variables t', x', υ' (and u and F) to be dimensionless, it is necessary to separate and multiply x and υ by the characteristic parameter L, which has the dimension of the quantity x. The transformation of the argument t to the dimensionless t' then has the form $t' = (P_0/mL)^{1/2}t$.

We represent problem (1.1) in the following form (the primes are omitted in order to simplify the notation)

$$\begin{aligned}&\dot{x} = \upsilon, \quad \dot{\upsilon} = u + F; \quad x(0) = x^0, \quad \upsilon(0) = \upsilon^0\\&x(T) = 0; \quad J[u] = T \to \min_u, \quad |u| \le 1, \quad |F| < 1\end{aligned} \tag{1.2}$$

To solve the problem, we apply optimality conditions in the form of the maximum principle.[1] Note that a special case of problem (1.2) and its modification when $F = 0$ have been considered earlier (see Refs. 1–3, etc.). The investigation of the optimal motion in the formulation (1.2) is of interest in problems of mechanics of the controlled flight of aircraft and spacecraft, including vehicles with a solar sail[4,5] and other types of engines, which do not lead to a significant change in the mass,[4] for special problems of approach, homing, target interception, reorientation, etc. In these investigations, modification of the terminal conditions for the position vector x and the velocity vector υ, in particular, arrival onto the boundary of a cylindrical domain or onto a specified manifold of smaller dimension, including when account is taken of constraints on the final velocity (see Refs. 6–9, etc.), can be of considerable importance.

We note that, in the general situation of non-coplanar vectors x^0, υ^0, F when $n \ge 3$, the dimension of problem (1.2) reduce to the case when $n = 3$. Actually, as a consequence of the central symmetry of the constraint imposed on the control, the optimal motion will occur in the three-dimensional Euclidean space E^3 containing the vectors x^0, υ^0, F. The place containing the vectors x^0, υ^0 can be taken as one of the coordinate planes and an appropriate rectangular system of coordinates can be introduced; the third axis of the system of coordinates in E^3 is added in the standard manner. If irregular small perturbations are possible, the system of coordinates must be reconstructed. In degenerate situations, the optimal motion can be planar when $([x^0 \times \upsilon^0] \cdot F) = 0$ or one-dimensional: the vectors x^0, υ^0, F are collinear or anticollinear. The solution of problem (1.2) is then constructed and investigated in general form and the property $n \le 3$ is not used.

2. The boundary value problem of the maximum principle

We introduce the vectors p and q, which are conjugate to x and υ respectively and construct the Hamiltonian function in the standard way.[1–3] From the conditions of the maximum principle, we find the expressions

$$p = \text{const}, \quad q = p(T-t), \quad u^* = q|q|^{-1} = p|p|^{-1} = \text{const} \tag{2.1}$$

Substitution of the control u^* (2.1) into the equations of motion (1.2) and elementary integration lead to parabolic trajectories of the form

$$\begin{aligned}&x = R(t, x^0, \upsilon^0) + S(t, F) + \xi t^2/2, \quad \xi = p|p|^{-1}, \quad \upsilon = \upsilon^0 + \xi t\\&R = x^0 + \upsilon^0 t, \quad S = Ft^2/2, \quad 0 \le t \le T, \quad T > 0\end{aligned} \tag{2.2}$$

The first term in the expressions for x (2.2) is the inertial translation of the point mass and the second term is the translation under the action of the known perturbation F. The third term contains the unknown unit vector ξ which, in the case of an unknown value $t = T$, satisfies the null final condition for x according to (1.2). To determine f ξ and T ($n+1$ unknown quantities), we have $n+1$ relations and the optimality condition

$$\begin{aligned}&\xi T^2/2 = -R(T, x^0, \upsilon^0) - S(T, F), \quad |\xi| = 1\\&\min_i T_i \to T^* = T^*(x^0, \upsilon^0, F), \quad \xi^* = \xi^*(x^0, \upsilon^0, F)\end{aligned} \tag{2.3}$$

As shown in Refs. 2 and 3, direct solution of the non-linear system of equations and the choice of the optimal root T^*, ξ^* is not possible either in the situation of a general position of the non-coplanar vectors x^0, υ^0, F or for cases which are close to degenerate. However, it follows from relations (2.3) that, if the optimal response time T^* can be successfully determined, then the required unit vector ξ^* is very easily found from the linear equation

$$\xi^* = -2R(T^*, x^0, \upsilon^0)/T^{*2} - F; \quad T^* = T^*(x^0, \upsilon^0, F) \tag{2.4}$$

which has an explicit mechanical content.

The required solution of the problem therefore essentially reduces to calculating of the quantity T^*. For this purpose, we use a previously developed procedure.[3] We take the square of equality (2.3) (or (2.4)) and obtain a fourth-order equation in T which is conveniently represented in the form

$$(1-f^2)T^4/4 = r^2(T,l,h,c) + fld_xT^2 + fhd_\upsilon T^3$$
$$r^2 = l^2 + 2clhT + h^2T^2, \quad l = |x^0|, \quad h = |\upsilon^0|, \quad c = \cos(x^0, \upsilon^0) = (x^0, \upsilon^0)l^{-1}h^{-1}$$
$$r = |R|, \quad f = |F|, \quad d_x = \cos(x^0, F) = (x^0, F)l^{-1}f^{-1}$$
$$d_\upsilon = \cos(\upsilon^0, F) = (\upsilon^0, F)h^{-1}f^{-1} \tag{2.5}$$

It follows from Eq. (2.5) that the required value of the response time T^* is actually determined by six parameters and not $3n$ ($n=3$). They vary within a permissible range: $l, h \geq 0$; $1 > f \geq 0$; $|c|$, $|d_{x,\upsilon}| \leq 1$. Note that, in the case of the planar problem ($n=2$), the parameters c, $d_{x,\upsilon}$ are connected by the relation $d_\upsilon = cd_x \pm ss_x$.

Owing to the negativity of the free term, the equation admits of one (T_1) or three ($T_{1,2,3}$) positive roots (the existence of multiple roots is not excluded[3]), from which the smallest root is chosen. Analysis indicates that the dependence of the required minimum value of T_i on the above parameters will not be smooth.[1,3] Moreover, all the solutions T_i, ξ_i satisfy the maximum principle relations, that is, fictitious solutions (of the type $r^2=1$, $r_{1,2}=\pm 1$) do not appear when expression (2.3) is squared.

For arbitrary fixed vectors x^0, υ^0, F, the minimum value of T^* can be found with the required accuracy numerically. The corresponding optimal open-loop control $u^* = \xi^*$ is determined after substituting T^* into Eq. (2.4). However, this control process may be ineffective when there are additional irregular perturbations and it is required to construct a feedback control. In this situation, it is necessary to implement the procedure for calculating T^* and ξ^*, as described above, on the basis of the current measured values of x, υ and F.

The analytical solution of Eq. (2.5) can be represented using the "Cardano formulae" (obtained for the first time by Ferrari). However, they are not very useful when analysing roots as a function of the parameters owing to the complexity the expressions for T_i.

3. Partial analysis of the solution

A procedure, associated with the construction of an explicit inverse relation,[3] can be used in the theoretical investigation of the dependence of T^* on any of the parameters of interest. It follows from Eq. (2.5) that the expressions for l, h and f are quadratic and those for c, $d_{x,\upsilon}$ are linear. Solving the corresponding equations, we obtain the required dependence of the parameters being analysed on the variable $T>0$. The values of the remaining parameters are assumed to be fixed during this procedure.

Note that the number of essential parameters in Eq. (2.5) can be reduced by one. In fact, according to the formulation of problem (1.2), the quantity $l>0$ and, when $l\sim 1$, it is convenient to normalize the parameters T and h as follows[3]: $\Theta = Tl^{-1/2}$, $\eta = hl^{-1/2}$ and the parameters f, c, $d_{x,\upsilon}$ remain unchanged. As a result, the equation reduces to the form (2.5) in which $l=1$.

When $f=0$, the solution of the equation in the form of the family of curves $\eta(\Theta, c)$ has been completely constructed and investigated for arbitrary $\Theta>0$, $|c|\leq 1$.[3] In particular, the existence of a critical value $c* = -\sqrt{8}/3$ has been established such that (see Section 6), when $c^* \geq c > -1$, three roots Θ_i are possible and there are discontinuities in the Bellman function

$$T^* = \min_i \Theta_i(\sqrt{l}h, c)\sqrt{l}, \quad i = 1,2,3$$

Using this procedure, it is possible to take into account the perturbation F, that is, the parameters $f>0$, $|d_{x,\upsilon}|\leq 1$. As a result, the required dependence is obtained in the permissible domain, defined by the relations

$$\eta = \eta_{1,2}(\Theta; c, f, d_x, d_\upsilon) \geq 0, \ \Theta > 0, \ 1 > f \geq 0, \ |c| \leq 1, \ |d_{x,\upsilon}| \leq 1$$
$$\eta_{1,2} = (-b \pm \sqrt{D})\Theta^{-2}/2, \ b = 2c\Theta + fd_\upsilon\Theta^3 \geq 0, \ D \geq 0$$
$$D = b^2 - 4\Theta^2(1 + fd_x\Theta^2) + (1-f^2)\Theta^6 \ (h = \sqrt{l}\eta, \ T = \sqrt{l}\Theta) \tag{3.1}$$

When $f \ll 1$, the quantity c, determining the property of the solution and the parameter of the family of curves (3.1) is important. For $0 \leq c \leq 1$, there is a one-to-one relation between η and Θ. When $0 > c > c^*$, segments of a two-valued dependence of η on Θ appear for which the values of Θ are uniquely determined from η. Hence, far from the critical value $c^*(c>c^*)$, a weak effect of the magnitude of f on the time-optimal values Θ would be expected, depending on the modulus of the velocity $h(l\sim 1)$. In the narrow domain $c^* \geq c > -1$, there is a non-unique relation between η and Θ which leads to discontinuities (jumps) in the value of Θ. The existence of a perturbation F can lead to a substantial rearrangement of the above-mentioned functional relation. It can facilitate or hinder the control process and, in particular, $F=f\xi$ or $F=-f\xi$ (see Section 4).

The characteristic qualitative properties of the relation between η and Θ which has been established are determined by the behaviour of the expression for D (3.1) which imposes considerable constraints on all the variable parameters. Note that the discriminant D is a cubic polynomial of the variable Θ^2, the free term of which is equal to zero. This property enables us to investigate the singularities of the function D easily and, in particular, its zeroes $\Theta_* > 0$ as a function of the parameters c, f, $d_{x,\upsilon}$:

$$\Theta_* = \sqrt{2}(1-f^2s_\upsilon^2)^{-1/2}(-f(cd_\upsilon - d_x) + \sqrt{\delta})^{1/2}, \quad \delta(c, f, d_x, d_\upsilon) \geq 0$$
$$\delta = f^2(cd_\upsilon - d_x)^2 + s^2(1 - f^2d_\upsilon^2), \quad s^2 = 1 - c^2, \quad s_\upsilon^2 = 1 - d_\upsilon^2 \tag{3.2}$$

It readily follows from relations (3.2) that the discriminant $\delta > 0$ when $c \neq \pm 1$, that is, in the regular case of non-coplanarity of the vectors x^0, $\upsilon 0$, F. In the degenerate case when $s = 0$, the quantity $\delta = 0$ since, when $c = \pm 1$, we have $d_x = \pm d_\upsilon$. For the values Θ_* (3.2), the quantity $D \equiv 0$ and the derivative of the function $\sqrt{D}$ (3.1) becomes unbounded when $\Theta \to \Theta_*$ for specified $c, f, d_{x,\upsilon}$. It also follows from the expression for $\eta_{1,2}$ (3.1) that the linear asymptotic expression for η as $\Theta \to \infty$ of the form

$$\eta = \eta_1 = (-fd_\upsilon + (1 - f^2 s_\upsilon^2)^{1/2})\Theta/2 + O(1) \to \infty \tag{3.3}$$

holds.

If $c \to -1$, the approximate representation

$$\eta = \eta_1 = 1/\Theta + O(1), \quad \Theta \ll 1, \quad 1 + c \ll \Theta \tag{3.4}$$

holds for the function η in the case of small values of Θ.

Formulae (3.1) and (3.2) enable us to investigate the required dependence of η on Θ and the other parameters fairly completely. The optimal response time T is thereby determined as a function of the variables l and h and fixed $c, f, d_{x,\upsilon}$.

4. Extremal conditions of motion

We will investigate how the optimal response time T depends on the perturbation vector F, that is, on the magnitude of the modulus f and the direction cosines d_x, d_υ. We will use formulae (2.2) and (2.5) with the aim of determining the extremal values of F^*, leading to the maximum and minimum values of T^*.

It is necessary to use the generalized procedure of the method of Lagrange multipliers in order to find the extrema of the implicit function T of F in the bounded (spherical) domain

$$T(F) \to \text{extr}_F, \quad |F|^2 \le f^2; \quad T = \text{Arg}_T \Phi(T, F)$$
$$\Phi \equiv -T^4/4 + r^2 + 2(R, S) + S^2, \quad S = FT^2/2 \tag{4.1}$$

The functions r and R are determined from to relations (2.3) and (2.5) (they are taken for $t = T$).

In order to solve problem (4.1), we will assume that the implicit function $T(F)$ has been determined and we will apply the method of Lagrange multipliers to it, taking into account the expression for the derivative (there are no internal extrema)

$$T'(F) + 2\lambda F = 0, \quad F^2 = f^2; \quad T' = -\Phi'_F/\Phi'_T$$

As a result of eliminating the Lagrange multiplier λ, the closed system of the $n + 1$ equations for finding the required solution $F^*_\pm$, $T^*_\pm$ is obtained. In fact, we have

$$\Phi(T, F) = 0, \quad F^2 = f^2, \quad \Phi'_F|\Phi'_F|^{-1} = \pm F/f$$
$$F^*_\pm(T) = \pm f r^{-1} R, \; R = x^0 + \upsilon^0 T, \; r = (l^2 + 2clhT + h^2T^2)^{-1/2}$$
$$T^*_\pm = \text{Arg}_T \Phi(T, F^*_\pm(T)) \tag{4.2}$$

Note that the third (vector) relation is degenerate. According to relations (4.2), the extremal vector $F^*_\pm$ is collinear (or anticollinear) to the displacement vector R of the point in the case of inertial motion. It lies in the plane of the vectors x^0, υ^0, and the motion is planar. The values of $F^*_\pm$ are still unknown since the required time $T^*_\pm$ has not been determined.

Substituting the expressions for $F^*_\pm$ (4.2) into the expression for $\Phi(T, F)$ and taking the square we obtain an equation of the eighth degree in T. Actually, the following relations hold

$$(1 - f^2)T^4/4 = r^2 \pm frT^2, \quad r = (l^2 + 2clhT + h^2T^2)^{1/2}$$

The standard procedure involves separating the term $\pm frT^2$ and squaring the equality. As a result, the above-mentioned eighth-degree equation is obtained for finding the optimal values T^*, taking into account of the extremal nature of the perturbation $F^*_\pm$. We now use the structure of the equations with respect to r and use a similar technique to construct the "inverse" function $r > 0$ as a function of the unknown T (and the parameter f). Solving the quadratic equations in r, we obtain relations for determining the optimal values $T^*_\pm$ for the extremal $F^*_\pm$

$$r_\pm = T^2(1 \mp f)/2 = (l^2 + 2clhT + h^2T^2)^{1/2}$$
$$\Theta^4/4 = 1 + 2c\eta\Theta + \eta^2\Theta^2, \quad \Theta = Tl^{-1/2}(1 \mp f)^{1/2}, \quad \eta = hl^{-1/2}(1 \mp f)^{-1/2} \tag{4.3}$$

According to relations (4.3), the extremal perturbations $F^*_\pm$ lead to a reduction or an increase in the modulus of the effective control.

By making the substitution $T \to \Theta$, $h \to \eta$, the equation for the required quantity T is to the form of the equation for the unknown quantity Θ, which is analogous to the case when $F = 0$ (when there is no perturbation). At the same time, the modulus of the velocity vector h is transformed to the quantity η, with the parameter c remaining unchanged. The solutions of the equation in $\Theta = \Theta(\eta, c)$ are fully investigated using the construction of the family of solutions $\eta = \eta(\Theta, c)$ (Ref. 3):

$$\eta = (-2c\Theta \pm \sqrt{\Delta})\Theta^{-2}/2; \quad \Delta = \Theta^6 - 4s^2\Theta^2 \ge 0 \tag{4.4}$$

The critical value $c_* = -\sqrt{8}/3$ turns out to be equal to that established earlier for the case when $F = 0$, which is an interesting qualitative effect of the solution of the time-optimal problem. This property does not hold in the general case when $F \neq F^*_\pm$. After determining the required value of Θ from to (4.4), using the given values of l, h, c and f, the extremal optimal response time is calculated and the corresponding control

$$T^*_\pm = \Theta^*(hl^{-1/2}(1 \mp f)^{-1/2}, c)l^{1/2}(1 \mp f)^{-1/2}$$
$$u^* = \xi^*_\pm = -2\frac{x^0 + v^0 T^*_\pm}{(1 \mp f)(T^*_\pm)^2} = -2\frac{R_\pm r_\pm(l, h, c, T^*_\pm)}{r_\pm (1 \mp f)(T^*_\pm)^2} \qquad (4.5)$$

is determined.

It can be seen that, in extremal cases, the optimal control u^* lies in the plane of the vectors x^0, v^0. Hence, the vectors x^0, v^0, $F^*_\pm$, $\xi^*_\pm$ turn out to be coplanar and the control problem is planar ($n = 2$). The two-dimensional time-optimal problem is therefore of interest a prior when analysing the effect of extremal perturbations.

5. Approximate analytical investigation

A local analysis of the required solution in the neighbourhood of the critical values of the parameters l, h and c in the case of the initial Eq. (2.5) is of interest for theory and applications.

1°. We will consider the case of small values of the velocity v^0: $h \ll 1$. Then, when $l \sim 1$, it is convenient to introduce the normalized quantities η and Θ and to represent Eq. (2.5) in the form

$$(1 - f^2)\Theta^4/4 = 1 + 2c\eta\Theta + \eta^2\Theta^2 + d_x f\Theta^2 + d_v f\eta\Theta^3; \quad \Theta = Tl^{-1/2}; \quad \eta = hl^{-1/2} \ll 1 \qquad (5.1)$$

The solution is investigated by the method of perturbations of a small parameter on the basis of the generating root (when $\eta = 0$) of the corresponding biquadratic equation

$$\Theta^*_0 = \sqrt{2}(1 - f^2)^{-1/2}(d_x f + (1 - s_x^2 f^2)^{1/2})^{1/2} > 0$$
$$d_x = \pm 1, \quad \Theta^+_0 \geq \Theta^*_0 \geq \Theta^-_0, \quad \Theta^\pm_0 = \sqrt{2}(1 \mp f)^{1/2} \qquad (5.2)$$

It is obvious that the unperturbed optimal motion is planar (in the plane of the vectors x^0, F), the extremal perturbation $F^*_\pm$ is collinear or anticollinear with the vector x^0, and the corresponding optimal motion is one-dimensional. Since the root Θ_0 (5.2) always turns out to be simple, the perturbed solution of Eq. (5.1) is constructed in powers of the parameter η and, in particular,

$$\Theta^* = \Theta^*_0 + \eta\Theta^*_1 + O(\eta^2)$$
$$\Theta^*_1 = (2c + d_v f\Theta^{*2}_0)((1 - f^2)\Theta^{*2}_0 - 2d_x f)^{-1} \qquad (5.3)$$

The denominator in the expression for Θ^*_1 is equal to $2(1 - s_x^2 f^2) > 0$. The subsequent approximations are extremely unwieldy and unsuitable for calculation and analysis.

2°. The opposite situation when $\eta \gg 1 (\eta \to \infty)$ is of definite interest both from theoretical and applied aspects. In this case, there are two limiting modes of motion: the basic mode for which $\Theta \sim \eta \to \infty$ when $c > -1$ and a critical mode for which $\Theta \sim \eta^{-1} \to 0$ when $c \approx -1$. In the case of the first more general mode of motion, that is, when there is a significant departure from the target or a close "transit" past the target ($x = 0$), the singularly perturbed equation (5.1) holds, which leads to the asymptotic solution

$$\Theta^* = \Theta^*_{-1}\eta + \Theta^*_0 + \Theta^*_1\eta^{-1} + \Theta^*_2\eta^{-2} + \ldots, \quad \eta^{-1} \ll 1$$
$$\Theta^*_{-1} = 2(d_v f + (1 - s_v^2 f^2)^{1/2})(1 - f^2)^{-1} > 0, \quad \Theta^*_0 = 0$$
$$\Theta^*_1 = (2c + d_x f\Theta^*_{-1})[(1 - f^2)\Theta^{*2}_{-1} - 2 - 3d_v f\Theta^*_{-1}]^{-1} \qquad (5.4)$$

The successive coefficients $\Theta_2, \Theta_3, \ldots$ of the expansion (5.4) are uniquely found in a similar way. The main content of the dependence which has been constructed is included in the first two terms Θ^*_{-1}, Θ^*_0. It is essential that $\Theta^*_{-1} > 0$ and $\Theta^*_0 = 0$; here the leading terms of the asymptotic expansion are independent of the parameters c and d_x. Note that, when $c = -1 + O(\eta^{-2})$, modes are possible for which $\Theta^* = \eta^{-1} + O(\eta^{-2})$ (the velocity vector v^0 is directed "exactly" opposite to the radius vector x^0).

3°. Suppose the initial point x^0 is located in a small neighbourhood of the terminal point $x(T) = 0$, that is, $l \ll 1$, but $h \sim 1$. The situation is similar to that considered above for $h \gg 1$, since it leads to the analysis of two types of motions: $c > -1$ and $c \approx -1$. We will investigate the more general case and normalize by h: we obtain the equation and its approximate solution

$$(1-f^2)\Theta^4/4 = \zeta^2 + 2c\zeta\Theta + \Theta^2 + d_x f\zeta\Theta^2 + d_\upsilon f\Theta^3$$

$$\Theta = Th^{-1}, \quad \zeta = lh^{-2} \ll 1, \quad -1 < c \le 1, \quad \Theta \sim 1$$

$$\Theta* = \Theta_0^* + \zeta\Theta_1^* + \zeta^2\Theta_2^* + \ldots, \ \Theta_0^* = 2(d_\upsilon f_\upsilon + (1-s_\upsilon^2 f^2)^{1/2})(1-f^2)^{-1} > 0$$

$$\Theta_1^* = (2c + d_x f\Theta_0^*)[(1-f^2)\Theta_0^{*2} - 2 - 3d_\upsilon f\Theta_0^*]^{-1} \neq 0 \tag{5.5}$$

The subsequent coefficients $\Theta_2^*, \Theta_3^*, \ldots$ are found in a similar way. The quantity $\Theta_0^* > 0$, the simple root of the quadratic equation, is of considerable significance. In the calculations of Θ_1^*, Θ_2^*, the expression in the square brackets (the denominator) does not vanish. Note that the leading terms of the expansions Θ_0^* (5.5) and Θ_1^* (5.4) are identical. Analysis shows[3,6] that, when $c = -1 + O(\xi)$, the situation $\Theta^* \sim \xi$ is impossible. If $c = -1 + O(\xi^2)$, this solution, which corresponds to homing without missing, is possible but, in this case, additional analysis of the conditions for Eq. (5.4) to be solvable in the form of expansions in fractional powers of ζ is required (the generating solution $\Theta_0^* = 0$ is a double solution, that is, degenerate). For example, when $c = -1 + \gamma\zeta^2$, $\gamma > 0$, there is the possibility of a "splitting" of the required root into a quantity of the order of ζ^2, which follows from the relations

$$\Theta* = \zeta\Theta_1^* + \zeta^2\Theta_2^* + \ldots, \quad \Theta_1 = 1$$

The quadratic equation

$$\Theta_2^2 - d_x f\Theta_2 - (1-f^2)/4 + 2\gamma + \kappa f = 0, \quad d_\upsilon = -d_x + \kappa\zeta$$

holds for the unknown Θ_2.

Verification of the non-negativity of the discriminant is required which is carried out in an elementary manner. The subsequent investigation in relation to the multiplicity of the root Θ_2^* is carried out using the theory of implicit functions.[10]

4°. Now suppose the initial point x^0 is located "asymptotically far" from the required terminal position $x(T) = 0$, that is, $\zeta \gg 1$ in Eq. (5.5). For the approximate solution of this equation it is natural to make the substitution $\Theta = \theta\sqrt{\xi}$ and to arrive at an equation which is identical to (5.1). In fact, on introducing the small parameter $\varepsilon = \xi^{-1/2} \ll 1$, we obtain relations of the form of (5.1)–(5.3)

$$(1-f^2)\theta^4/4 = 1 + 2\varepsilon c\theta + \varepsilon^2\theta^2 + d_x f\theta^2 + \varepsilon d_\upsilon f\theta^3$$

$$\theta* = \theta_0^* + \varepsilon\theta_1^* + \varepsilon^2\theta_2^* + \ldots, \ \theta_0^* = \sqrt{2}(1-f^2)^{-1/2}(d_x f + (1-s_x^2 f^2)^{1/2})^{1/2} > 0$$

$$\theta_1^* = (2c + d_\upsilon f\theta_0^{*2})((1-f^2)\theta_0^{*2} - 2d_x f)^{-1} \tag{5.6}$$

The root θ_0^* of the biquadratic equation (5.6) when $\varepsilon = 0$ is simple. The subsequent coefficients $\theta_2^*, \theta_3^*, \ldots$ are found uniquely from linear relations. Since the initial value $\Theta^* = \theta^*/\varepsilon \sim \varepsilon^{-1}$, the basic asymptotic when $l \to \infty$ is determined by the coefficients θ_0^*, θ_1^* (5.6). Note that, in the leading term θ_0^*, there is no dependence on c, d_υ; it appears in θ_1^*, that is, in the terms of the order of unity for Θ^*.

Hence, in Sections 2–5, a fairly detailed analytical investigation of the problem of the optimal speed of response has been carried out for the case of a stationary perturbation.

6. The time optimal motion response for a variable perturbation

We will briefly investigate a modified problem of the type of (1.2) in the case of a bounded, integrable and, in particular, continuous perturbation $F(t)$:

$$\dot{\upsilon} = u + F(t), \quad \dot{x} = \upsilon; \quad \upsilon(t_0) = \upsilon^0, \quad x(t_0) = x^0$$

$$x(T) = 0, \quad \Delta T = T - t_0 \to \min_u, \quad |u| \le 1, \quad |F| \le f_0 < 1 \tag{6.1}$$

We will use a solution procedure based on the necessary conditions for optimality in the form of the maximum principle.[1] Expressions are obtained for the optimal control and the phase trajectory

$$u* = \xi = \text{const}, \quad \upsilon = \upsilon^0 + \xi\Delta t + V(\Delta t, t_0), \ V = \int_0^{\Delta t} F(t_0+\tau)d\tau$$

$$x = R(x^0, \upsilon^0, \Delta t) + \xi\Delta t^2/2 + S(\Delta t, t_0), \ S = \int_0^{\Delta t} (\Delta t - \tau)F(t_0+\tau)d\tau$$

$$\Delta t = t - t_0, \quad R = x^0 + \upsilon^0\Delta t, \quad 0 \le \Delta t \le \Delta T \tag{6.2}$$

in a similar way to that explained in Section 2.

In expressions (6.2), the constant unit vector ξ and the quantity of T, that is, the interval ΔT, are unknown. These unknowns are determined by solving a non-linear system of equations similar to (2.3)

$$R(x^0, \upsilon^0, \Delta T) + \xi\Delta T^2/2 + S(\Delta T, t_0) = 0$$
$$\xi = \xi^*(x^0, \upsilon^0, t_0), \quad \xi^2 = 1, \quad T^* = \Delta T^*(x^0, \upsilon^0, t_0) + t_0 \tag{6.3}$$

As previously, in the case of a known value of ΔT^*, the required unit vector ξ^* is found from the linear equation (6.3) in an elementary way. The main difficulty is in calculating of the minimum interval of the optimal of response time ΔT^* for the specified t_0, x^0, υ^0 and $F(t)$. In order to find it, it is necessary to solve the scalar equation

$$1/4T^4 = R^2 + 2(R, S) + S^2, \quad \Delta T_i \to \min_i, \quad \Delta T^* > 0 \tag{6.4}$$

which is analogous to Eq. (2.5). The dependence of S on ΔT can be extremely complicated.

By virtue of the estimates of the function R for $l > 0$ and of the function S for $\Delta T > 0$ of the form

$$R^2 = l^2 > 0, \quad \Delta T = 0; \quad |S| \le f_0\Delta T^2/2, \quad \Delta T > 0$$

it follows that Eq. (6.4) has a positive root ΔT_i for arbitrary x^0, υ^0 and functions $F(t)$ with the boundedness condition (6.1). The minimum value ΔT^* is simply obtained by a numerical double-sweep method using this parameter and using a refining procedure (the methods of secants, tangents, etc). In particular, if the value of f_0 is sufficiently small, then a highly accurate approximate solution is obtained using any numerical successive-approximation procedure or analytically by expansions in integral or fractional powers of the parameter $f_0 \ll 1$.

They are based on the generating solution for Eq. (6.4) when $s = 0$: $\Delta T^4/4 = R^2$, which has been investigated in detail.[3] Note that the unperturbed root ΔT_0^* can be either simple or degenerate (a double root). The existence of multiple roots is proved by an analysis of the compatibility of Eq. (6.4) (see above) and the derivative with respect to ΔT: $\Delta T^3 = 2h(cl + h\Delta T)$ when $F \equiv 0$ (that is, when $S = 0$). By means of simple transformations, these relations lead to the problem of the compatibility of the two equations in the domain of permissible values of θ, η and c:

$$2 + 3c\eta\theta + \eta^2\theta^2 = 0, \quad \theta^4/4 + 1 + c\eta\theta = 0$$
$$\theta > 0, \quad \eta > 0, \quad |c| \le 1 \; (\theta = \Delta T l^{-1/2}, \; \eta = hl^{-1/2}) \tag{6.5}$$

We solve the first equation of (6.5) for the unknown θ and write out the condition $\theta > 0$ and the condition for the compatibility of the second relation

$$\theta_0^* = \theta_{1,2} = (-3c \pm \sqrt{9c^2 - 8})\eta^{-1}/2, \; -1 \le c \le -\sqrt{8}/3 = c^*$$
$$1 + \theta_{1,2}^4/4 = -c\eta\theta_{1,2}, \quad 4/3 \le -c\eta\theta_1 \le 2, \quad 1 \le -c\eta\theta_2 \le 4/3 \tag{6.6}$$

For sufficiently large values of η, the magnitudes of $\theta_{1,2}$ will be as small as desired and the product $\eta\theta_0^*$ will be bounded and such that the constraints on $-c\eta\theta_0^*$ (6.6) will be satisfied. In the case of a specified value of c, $-1 \le c \le c^*$, the equalities hold for the conditions

$$\eta_{1,2} = 2^{-3/2}(-3c \pm \sqrt{9c^2 - 8})[1/2(3c^2 \mp c\sqrt{9c^2 - 8}) - 1]^{-1/4} \tag{6.7}$$

Note that $\eta_2 \to \infty$ when $c \to -1$, since the expression for η_2 in the square brackets in (6.7) tends to zero; the minimum value of $\eta = \eta_1 = \sqrt{2}$. Hence, double roots θ_0^* are possible for values of c from the interval (6.6), which makes it difficult to apply standard iterative procedures to them when $f_0 \ll 1$. Generally speaking, expansions in integral powers f_k^0 are unsound and expansions of the required solution in fractional powers $f_0^{k/2}$ hold. This procedure requires additional analysis of the solvability conditions, that is, of the existence of the perturbed roots in the neighbourhood of degenerate generating values. Note that a triple root $\theta_0^* = (4/3)^{1/4}$ is possible when $c = c^* = -\sqrt{8}/3$, $\eta = 3^{1/4}$.

The situation when the values of the vector F_n belong to a fixed subspace $E^{n'} \subset E^n$, $2 \le n' < n$ can be considered together with the case of an arbitrary bounded vector function $F(t) \in E^n$ which has been investigated above. The optimal motion of the controlled system then occurs in a geometric space of dimension n'', $n' \le n'' \le n$. In particular, the direction of the vector F can be fixed in the space: $F(t) = \nu f(t)$, where the vector $\nu \in E^n$ $\nu = \text{const}$, and $f(t)$ is a scalar function. Without loss in generality, we set $|\nu| = 1$. Then, as in Sections 1–5, the optimal controlled motion is obtained in the space E^3 formed by the vectors x^0, υ^0, ν. The structure of the functions V and S in formulae (6.2)–(6.4) will be somewhat simpler, since single and double integration of the single function $f(t)$ is required. Only the constant vectors x^0, υ^0, ξ, ν, which generate the parameters l, h, c, d_x, d_υ in Eq. (6.4), and the factor (quadrature) which depends on ΔT, t_0 and is analogous to the parameter $fT^2/2$ in Eq. (2.5) are subject to the geometric operations. The solution can be constructed numerically in the case of specific expressions for $F(t)$ and $f(t)$.

The formulation of the time-optimal problem for the time-varying system (6.1) depends very much initial instant t_0. The solution of the feedback control problem, that is, the Bellman function ΔT^*, the time of termination of the process T^* and the optimal control u^* therefore depend on the current phase coordinates x and υ and the time t

$$u^* = \xi^*(x, \upsilon, t), \ \Delta T^* = \Delta T^*(x, \upsilon, t),$$
$$T^* = t + \Delta T^*(x, \upsilon, t) \tag{6.8}$$

The minimum value of ΔT^* is calculated from (6.3) at the actual instant of time t for the measured values of x and υ. As was established above, the Bellman function ΔT^* (6.8) has jump discontinuities (and, together with it, the control u^*) in the case of fairly small perturbations in the domain $-1 \le c \le c^*$ and fairly large values of the quantity $hl^{-1/2}$.

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